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THE USE OF A STABLE BOUNDARY ANALOGUE OF THE COLLOCATION METHOD TO APPROXIMATE THE SOLUTION OF SOME PROBLEMS IN MECHANICS†

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To solve boundary-value problems for elliptic equations, the boundary analogue of the method of least squares is replaced by a boundary analogue of the collocation method. The change is made using a discrete representation of the scalar product in the spaces of functions which, in the case of a smooth boundary, are integrable with their square over the boundary of the region, and of functions which, in the case of a piecewise-linear boundary, are integrable with their square, when weighted, over the boundary of the region. The method used to choose the collocation points which ensure the collocation method to be stable is justified for the case of the Dirichlet problem. © 1998 Elsevier Science Ltd. All rights reserved.

The use of a boundary analogue of the method of least squares to find the coefficients of the expansion of an approximate solution of a boundary-value problem in series of powers of global basic functions was considered in [1, 2] and proved to be stable for the Dirichlet problem in regions with smooth and piecewise-linear boundaries.

This has a number of advantages over existing methods. It allows a reduction of one in the Euclidean dimension of the problem and does not require discretization of the region, which leads to a simpler algorithm and has advantages in the case of problems with changing boundaries. The method yields systems of linear algebraic equations with a well-conditioned matrix. Among its drawbacks is the fact that there has been little research on the relation between the rate of convergence and the smoothness of the boundary. The range of problems to which the method can be applied is restricted to boundary-value problems for linear elliptic differential equations with constant or polynomial coefficients.

The method devised in [3] combines an exact solution in blocks with an approximate solution found on the boundaries of blocks which cover a polygonal region.

1. THE BOUNDARY ANALOGUE OF THE METHOD OF LEAST SQUARES

The method used to solve boundary-value problems for elliptic differential equations in [1] involves expanding the solution in powers of global basis functions of the kernel of the differential operator. It leads to a boundary-value problem for a homogeneous differential equation with a corresponding non-homogeneous boundary condition

$$Lu=0,\quad x\in\Omega,\quad \Omega\subset\mathbb{R}^n;\quad lu|_\Gamma=h(y),\quad y\in\Gamma \tag{1.1}$$

where $\Gamma=\partial\Omega$ is a Lipschitz boundary, L is an elliptic differential operator with constant coefficients and l is a boundary condition operator. The approximate solution $u^{(N)}$ of problem (1.1) is sought in the form

$$u^{(N)}(x)=\sum_{k=1}^{N}a_k^{(N)}\psi_k(x),\quad x\in\Omega,\quad \psi_k(x)\in\ker L \tag{1.2}$$

Here and in Sections 1 and 2 below $k=1,\dots,N$.

The coefficients $a_k^{(N)}$ can be found using a boundary analogue of the method of least squares (BAMLS) [4] from the condition

$$\min_{a_1,a_2,\dots,a_N}\|u^{(N)}-h\|^2_{V(\Gamma)} \tag{1.3}$$

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Here $V(\Gamma)$ is a real Hilbert space in which the norm is generated by the scalar product $(\cdot, \cdot)_{V(\Gamma)}$.
Condition (1.3) leads to a system of linear algebraic equations

$$\mathbf{G}^{(N)}\mathbf{a}^{(N)} = \mathbf{h}^{(N)} \tag{1.4}$$

where the vectors $\mathbf{a}^{(N)}$, $\mathbf{h}^{(N)}$ have elements $a_k^{(N)}$, $h_k^{(N)} = (h, l\psi_k)_{V(\Gamma)}$, respectively, and $\mathbf{G}^{(N)}$ is the Gram matrix of the first N elements of the system $\Psi = \{\psi_1, \dots, \psi_N, \dots\}$ in Hilbert space with the scalar product $(l\cdot, l\cdot)_{V(\Gamma)}$.

In most cases the system Ψ turns out to be non-orthogonal, and so the main issue is the stability of the method. The stability of the BAMLS, as defined in [5], depends only on the properties of the system Ψ in the corresponding Hilbert space. A necessary and sufficient condition for the BAMLS to be stable is that the system Ψ must be strongly minimal in a space with scalar product $(l\cdot, l\cdot)_{V(\Gamma)}$, or in other words that there should be a positive number λ_0 for which

$$\lambda_0 < \lambda_{\min}(\mathbf{G}^{(N)}), \quad \forall N \in \mathbb{N}$$

where $\lambda_{\min}(\mathbf{G}^{(N)})$ is the smallest eigenvalue of the matrix $\mathbf{G}^{(N)}$ [5].

Remark 1. If L is an elliptic differential operator with polynomial coefficients, a system of polynomials which satisfy a homogeneous differential equation can be constructed.

2. THE TRANSITION TO THE BOUNDARY ANALOGUE OF THE COLLOCATION METHOD

Let $\Omega \subset \mathbb{R}^2$, $\Gamma = \partial\Omega$, $V(\Gamma) = L_2(\Gamma)$. We will consider the transition from the BAMLS to a boundary analogue of the collocation method based on the use of quadrature formulae. The elements of the matrix $\mathbf{G}^{(N)}$ are defined by the expression

$$G_{ij}^{(N)} = (l\psi_i, l\psi_j)_{2,\Gamma} = \int_0^1 l\psi_i(x_1(s),\ x_2(s))l\psi_j(x_1(s),\ x_2(s))d\Gamma(s) \tag{2.1}$$

Here and everywhere below, unless otherwise stated, $i, j = 1, \dots, N$.

Let $d\Gamma(s) = \gamma(s)ds$, $\gamma(s) \in C^\infty([0, 1])$, $l\psi_i(x_1(s)) \in C^\infty([0, 1])$. Applying the Gauss quadrature formula with N nodes $s_k^{(N)} \in [0, 1]$ and coefficients $A_k^{(N)}$ to (2.1), we obtain

$$(l\psi_i, l\psi_j)_{2,\Gamma} = \sum_{k=1}^N A_k^{(N)} l\psi_i(x_1(s_k^{(N)}),\ x_2(s_k^{(N)}))l\psi_j(x_1(s_k^{(N)}),\ x_2(s_k^{(N)}))\gamma(s_k^{(N)}) + \delta B_{ij}^{(N)} \tag{2.2}$$

where $\delta B_{ij}^{(N)}$ is the error of the quadrature formula.
We now introduce the notation

$$x_{1k}^{(N)} = x_1(s_k^{(N)}), \quad x_{2k}^{(N)} = x_2(s_k^{(N)}), \quad y_k^{(N)} = y(s_k^{(N)}), \quad \gamma_k^{(N)} = \gamma(s_k^{(N)})$$

Then the system of linear equations of the method of least squares (1.4) can be put in the form

$$(\mathbf{B}^{(N)} + \delta\mathbf{B}^{(N)})\mathbf{a}^{(N)} = \mathbf{h}^{(N)} + \delta\mathbf{h}^{(N)} \tag{2.3}$$

where the elements of the matrix $\mathbf{B}^{(N)}$ and the vector $\mathbf{h}^{(N)}$ are

$$B_{ij}^{(N)} = \sum_{k=1}^N \sqrt{A_k^{(N)}\gamma_k^{(N)}} l\psi_i(x_{1k}^{(N)},\ x_{2k}^{(N)})\sqrt{A_k^{(N)}\gamma_k^{(N)}} l\psi_j(x_{1k}^{(N)},\ x_{2k}^{(N)})$$

$$h_i^{(N)} = \sum_{k=1}^N A_k^{(N)}\gamma_k^{(N)} l\psi_i(x_{1k}^{(N)},\ x_{2k}^{(N)})h(s_k^{(N)})$$

and $\delta h_i^{(N)}$ is the error of the quadrature formula in calculating the right-hand side of the system when the method of least squares is used. We note that the matrix $\mathbf{B}^{(N)}$ can be written in the form $\mathbf{B}^{(N)} = \mathbf{K}^{(N)}[\mathbf{K}^{(N)}]^T$, where the elements of the matrix $\mathbf{K}^{(N)}$ have the form

$$K_{ij}^{(N)} = \sqrt{A_j^{(N)}\gamma_j^{(N)}} l\psi_i(x_{1j}^{(N)},\ x_{2j}^{(N)}) \tag{2.4}$$

We will denote by $\mu(\mathbf{M})$ the condition number of the matrix $\mathbf{M}$ with respect to the second matrix norm: $\mu(\mathbf{M}) = \| \mathbf{M} \|_2 \| \mathbf{M}^{-1} \|_2$. We know [6] that

$$\mu(\mathbf{B}^{(N)}) = \mu^2([\mathbf{K}^{(N)}]^T), \quad \|([\mathbf{K}^{(N)}]^T)^{-1}\|_2 = \lambda_{\min}^{-1/2}(\mathbf{B}^{(N)}) \tag{2.5}$$

Let $\bar{\mathbf{a}}^{(N)}$ denote a solution of system (2.3) and let $\tilde{\mathbf{a}}^{(N)}$ denote a solution of the system

$$\mathbf{B}^{(N)}\mathbf{a}^{(N)} = \mathbf{h}^{(N)} \tag{2.6}$$

A theorem which gives the sufficient conditions for convergence can be obtained from the relation [5]

$$\|\bar{\mathbf{a}}^{(N)} - \tilde{\mathbf{a}}^{(N)}\|_2 \leq \frac{C\lambda_{\min}^{-\frac{3}{2}}(\mathbf{G}^{(N)})\|\delta\mathbf{B}^{(N)}\|_2 + \lambda_{\min}^{-1}(\mathbf{G}^{(N)})\|\delta\mathbf{h}^{(N)}\|_2}{1 - \lambda_{\min}^{-1}(\mathbf{G}^{(N)})\|\delta\mathbf{B}^{(N)}\|_2} \tag{2.7}$$

Theorem 1. Let the system Ψ and the quadrature formula used for the approximate computation of the integral (2.1) be such that

$$\lim_{N\to\infty} [\lambda_{\min}^{-3/2}(\mathbf{G}^{(N)})\|\delta\mathbf{B}^{(N)}\|_2] = 0, \quad \lim_{N\to\infty} [\lambda_{\min}^{-1}(\mathbf{G}^{(N)})\|\delta\mathbf{h}^{(N)}\|_2] = 0$$

Then $\lim_{N\to\infty} [\| \bar{\mathbf{a}}^{(N)} - \tilde{\mathbf{a}}^{(N)} \|_2 = 0$.

Remark 2. For strongly minimal systems, it is sufficient that

$$\lim_{N\to\infty}\left[N \max_{i,j=1,2,\ldots,N} |\delta B_{ij}^{(N)}|\right] = 0, \quad \lim_{N\to\infty}\left[N \max_{j=1,2,\ldots,N} |\delta h_j^{(N)}|\right] = 0 \tag{2.8}$$

This follows from the estimate $\| \cdot \|_2$.

The quantities $\delta B_{ij}^{(N)}$ and $\delta h_j^{(N)}$ are the errors in using the quadrature formulae for integrating functions of the form $l\psi_i(x_1(s), x_2(s))l\psi_j(x_1(s), x_2(s))\gamma(s)$ and $h(y(s))l\psi_j(x_1(s), x_2(s))\gamma(s)$, respectively over the segment [0, 1]. The rate of convergence of the quadrature formulae depends on the order of smoothness of the integrands. Thus, if [7]

$$\begin{aligned} &\frac{d}{ds}[l\psi_i(x_1(s),\ x_2(s))l\psi_j(x_1(s),\ x_2(s))\gamma(s)] \in \mathrm{Lip}([0,1]) \\ &\frac{d}{ds}[h(y(s))l\psi_j(x_1(s),\ x_2(s))\gamma(s)] \in \mathrm{Lip}([0,1]) \end{aligned} \tag{2.9}$$

where Lip([0, 1]) is the class of functions which have Lipschitz properties on [0, 1], conditions (2.8) are satisfied.

Remark 3. Condition (2.9) is satisfied if $\psi_1(x_1,x_2)$, $\psi_2(x_1,x_2), \ldots, h(y)$ and $\gamma(s)$ are infinitely-differentiable functions.

Consider the system of functionals

$$W_j(\psi) = \int_0^1 \psi(s)w_j(s)\gamma(s)ds \tag{2.10}$$

We will find the coefficients of series (1.2) from the condition [4]

$$W_j\left(\sum_{k=1}^{N} a_k^{(N)} l\psi_k - h\right) = 0 \tag{2.11}$$

We choose functions $w_j(s)$ in (2.10) for which (2.11) is a system of linear algebraic equations in $\mathbf{a}^{(N)}$ with matrix $\mathbf{K}^{(N)}$ defined by (2.4). We put

$$w_j^{(N)}(s) = \sqrt{A_j^{(N)} / \gamma(s)}\delta(s - s_j^{(N)}) \tag{2.12}$$

where $\delta(\cdot)$ is the delta function and $s_j^{(N)}$ are the nodes of the quadrature formula. Then system (2.11) can be written in the form

$$[\mathbf{K}^{(N)}]^T \mathbf{a}^{(N)} = \bar{\mathbf{h}}^{(N)} \tag{2.13}$$

The elements of the matrix $\mathbf{K}^{(N)}$ are defined by formula (2.4), and the elements of the vector $\bar{\mathbf{h}}^{(N)}$ are

$$\bar{\mathbf{h}}_j^{(N)} = h(y_j^{(N)})\sqrt{A_j^{(N)}\gamma_j^{(N)}} \tag{2.14}$$

The solution of the system for the collocation method (CM) (2.13) is the same as that of (2.6) and converges in a Euclidean norm, as $N \to \infty$, to the solution of (1.4). We have thus found basis functions for which the collocation method converges.

Remark 4. The matrices $\mathbf{B}^{(N)}$ and $\delta\mathbf{B}^{(N)}$ are Hermitian, and it thus follows from Weyl's theorem [6] that

$$|\lambda_{\max}(\mathbf{G}^{(N)}) - \lambda_{\max}(\mathbf{B}^{(N)})| \leqslant \|\delta\mathbf{B}^{(N)}\|_2, \quad |\lambda_{\min}(\mathbf{G}^{(N)}) - \lambda_{\min}(\mathbf{B}^{(N)})| \leqslant \|\delta\mathbf{B}^{(N)}\|_2$$

But as $\| \delta\mathbf{B}^{(N)} \|_2 \to 0$ from (2.5) we have

$$\lim_{N\to\infty} \mu([\mathbf{K}^{(N)}]^T) = \lim_{N\to\infty} \sqrt{\mu(\mathbf{G}^{(N)})}$$

Remark 5. If $n > 2$, the transition to the boundary analogue of the CM is made using cubature formulae. The results are similar to those for $n = 2$.

3. THE DIRICHLET PROBLEM IN REGIONS WITH SMOOTH BOUNDARIES

Let $\Delta u = 0$ and $\Omega \subset \mathbb{R}^2$, $\Gamma = \partial\Omega$, $u|_\Gamma = h(y), y \in \Gamma$. We take Ψ to be the system

$$\{1, \operatorname{Re} z, \operatorname{Im} z, \operatorname{Re} z^2, \operatorname{Im} z^2, \ldots\}, \quad z = x_1 + ix_2$$

The convergence of the BAMLS with respect to system (3.1) and its strong minimality are proved in [1]. Let the boundary of the region Ω be given parametrically: $\Gamma = \{(x_1, x_2)|x_1 = x_1(s), x_2 = x_2(s), s \in [0, 1]\}$. If the functions $x_1(s), x_2(s) \in C^\infty([0, 1])$, then the system of linear algebraic equations of the CM with weight functions (2.12) is solvable for any order of approximation, the CM converges, and Remark 4 regarding the condition number of the matrix of the system applies.

4. THE DIRICHLET PROBLEM IN REGIONS WITH PIECEWISE-LINEAR BOUNDARIES

We now consider the Dirichlet problem in the region $\Omega \subset \mathbb{R}^2$ with a piecewise-linear boundary Γ. In this case

$$\Gamma = \bigcup_{m=1}^{L} \Gamma_m \tag{4.1}$$

where $\Gamma_m = \{(x_1, x_2)|x_r = x_{rm}(s) \equiv a_{rm}s + b_{rm}, r = 1, 2, s \in [s_m, s_{m+1}]\}$.

We shall assume that $s \in [0, 1]$ and, therefore

$$\bigcup_{m=0}^{L-1} [s_m, s_{m+1}] = [0, 1]$$

Let $\pi\alpha_m$ be the interior angles of the region Ω at points $(x_1(s_m), x_2(s_m))$ $(m = 1, \ldots, L)$. We introduce a Hilbert space with weight $L_2(\Gamma; p)$, where

$$p(s) = \prod_{m=0}^{L} [(x_1(s) - x_1(s_m))^2 + (x_2(s) - x_2(s_m))^2]^{1/\alpha_m - 1}$$

In [1] we proved that system (3.1) is strongly minimal in $L_2(\Gamma; p)$, we considered how to use the BAMLS to find an approximate solution of the Dirichlet problem, and we gave an error estimate analogous to the Runge estimate for the scalar product computed over the boundary.

In order to construct the basis functions of the CM for which the system is solvable and the method converges, and which also give some indication of how the condition number of the matrix of the system of linear algebraic equations of the CM behaves, we write an expression for the scalar product in $L_2(\Gamma; p)$

$$(\psi_i,\psi_j)_{2,\Gamma;p}=\int_0^1 \psi_i(x_1(s),\ x_2(s))\psi_j(x_1(s),\ x_2(s))p(s)\gamma(s)ds=$$

$$=\sum_{m=0}^{L}\left\{\int_{s_m}^{\theta_{1m}} \psi_{ij}(x_{1m}(s),\ x_{2m}(s))p_{1m}(s)(s-s_m)^{1/\alpha_m-1}\gamma_m(s)ds+\right.$$

$$+\int_{\theta_{1m}}^{\theta_{2m}} \psi_{ij}(x_{1m}(s),\ x_{2m}(s))p_m(s)\gamma_m(s)ds+$$

$$\left.+\int_{\theta_{2m}}^{s_{m+1}} \psi_{ij}(x_{1m}(s),\ x_{2m}(s))p_m(s)(s_{m+1}-s)^{1/\alpha_{m+1}-1}\gamma_m(s)ds\right\} \tag{4.2}$$

$$p_m(s)=p(s),\quad s_m\leqslant s\leqslant s_{m+1},\quad p_{1m}(s)=p_m(s)/(s-s_m)^{1/\alpha_m-1}$$

$$p_{2m}(s)=p_m(s)/(s_{m+1}-s)^{1/\alpha_{m+1}-1}$$

$$\gamma_m(s)=\sqrt{a_{1m}^2+a_{2m}^2},\quad s_m<\theta_{1m}<\theta_{2m}<s_{m+1}$$

$$\psi_{ij}(x_{1m}(s),\ x_{2m}(s))=\psi_i(x_{1m}(s),\ x_{2m}(s))\psi_j(x_{1m}(s),\ x_{2m}(s))$$

The unweighted Gauss quadrature formula is used on $[\theta_{1m}, \theta_{2m}]$, and the Gauss quadrature formulae with weights $(s-s_m)^{1/\alpha_m-1}$ and $(s_{m+1}-s)^{1/\alpha_{m+1}-1}$, respectively, on $[s_m, \theta_{1m}]$ and $[\theta_{2m}, s_{m+1}]$. Then (4.2) can be written in the form

$$(\psi_i,\psi_j)_{2,\Gamma;p}=\sum_{m=0}^{L}\left\{\sum_{n=1}^{N_{1m}} C_{1n}^{N_{1m}}\psi_{ij}(x_{11mn},x_{21mn})p_{1m}(s_{1mn})\gamma_m(s_{1mn})+\right.$$

$$+\sum_{n=1}^{N_m} C_n^{N_m}\psi_{ij}(x_{1mn},x_{2mn})p_m(s_{mn})\gamma_m(s_{mn})+$$

$$\left.+\sum_{n=1}^{N_{2m}} C_{2n}^{N_{2m}}\psi_{ij}(x_{12mn},x_{22mn})p_{2m}(s_{2mn})\gamma_m(s_{2mn})+R_{1mij}^{N_{1m}}+R_{mij}^{N_m}+R_{2mij}^{N_{2m}}\right\} \tag{4.3}$$

where s_{1mn}, $C_{1n}^{N_{1m}}$ are the nodes and coefficients of the Gauss quadrature formula with weight $(s-s_m)^{1/\alpha_m-1}$ $(n = 1, \ldots, N_{1m})$, s_{mn}, $C_n^{N_m}$ are the nodes and coefficients of the unweighted formula $(n = 1, \ldots, N_m)$, and s_{2mn}, $C_{2n}^{N_{2m}}$ are the nodes and coefficients of the formula with weight $(s_{m+1}-s)^{1/\alpha_{m+1}-1}$ $(n = 1, \ldots, N_{2m})$, $x_{rqmn} = x_{rm}(s_{qmn})$ $(n = 1, \ldots, N_{qm})$, $x_{rmn} = x_{rm}(s_{mn})$ $(n = 1, \ldots, N_m)$; $r, q = 1, 2$, $m = 0, \ldots, L$, where $\Sigma_{m=0}^{L}(N_m + N_{1m} + N_{2m}) = N$, equal to the number of basis functions. We will refer to the points $x_{11mn}, x_{1mn}, x_{12mn}$ as x_{1k}, the points $x_{21mn}, x_{2mn}, x_{22mn}$ as x_{2k} and the points s_{1mn}, s_{mn}, s_{2mn} as s_k, and write Eq. (4.3) in the form

$$(\psi_i,\psi_j)_{2,\Gamma;p}=\sum_{k=1}^{N} D_k^{(N)}\psi_{ij}(x_{1k}^{(N)},x_{2k}^{(N)})P(s_k^{(N)})\gamma(s_k)+R_{ij}^{(N)}$$

$$P(s_k)=\begin{cases} p_{1m}(s_{1mn}), \text{ for } k \text{ corresponding to } (1,m,n)\\ p_m(s_{mn}), \text{ for } k \text{ corresponding to } (m,n)\\ p_{2m}(s_{2mn}), \text{ for } k \text{ corresponding to } (2,m,n)\end{cases}$$

$$D_k^{(N)}=\begin{cases} C_{1m}^{N_{1m}}, \text{ for } k \text{ corresponding to } (1,m,n)\\ C_m^{N_m}, \text{ for } k \text{ corresponding to } (m,n)\\ C_{2m}^{N_{2m}}, \text{ for } k \text{ corresponding to } (2,m,n)\end{cases}$$

$$R_{ij}^{(N)}=\sum_{m=0}^{L}[R_{1mij}^{N_{1m}}+R_{mij}^{N_m}+R_{2mij}^{N_{2m}}]$$

In this case, for the system of linear equations (2.3)

$$\mathbf{B}^{(N)} = \mathbf{K}^{(N)}[\mathbf{K}^{(N)}]^T$$
$$K_{ik}^{(N)} = \sqrt{D_k^{(N)}\gamma(s_k^{(N)})P(s_k^{(N)})}\psi_i(x_{1k}^{(N)}, x_{2k}^{(N)}) \tag{4.4}$$
$$h_i^{(N)} = \sum_{k=1}^{N} D_k^{(N)} h(y(s_k^{(N)}))\psi_i(x_{1k}^{(N)}, x_{2k}^{(N)})\gamma_k^{(N)}$$

and $\delta B_{ij}^{(N)} = R_{ij}^{(N)}$, $\delta h_i^{(N)}$ are the errors of the quadrature formulae. We select the weight functions in (2.10) as

$$w_j(s) = \sqrt{\frac{D_j^{(N)}P(s_j^{(N)})}{\gamma^{(N)}(s)}}\delta(s - s_j^{(N)}) \tag{4.5}$$

and obtain a system of linear equations (2.13) in which $\mathbf{K}^{(N)}$ is defined by formulae (4.4), and the elements of the vector of the right-hand side are given by the formula

$$\bar{h}_i^{(N)} = h(s_i^{(N)})\sqrt{D_i^{(N)}\gamma(s_i^{(N)})P(s_i^{(N)})}, \quad i = 1, \ldots, N$$

Let $\bar{\mathbf{a}}^{(N)}$, $\tilde{\mathbf{a}}^{(N)}$ be solutions of system (2.3), (2.6) for (4.4). We note that $\tilde{\mathbf{a}}^{(N)}$ is the same as solution (2.13) for (4.4). If in (2.7) we denote the Gram matrix of the first N elements of the system $\{\psi_i\}_{i=1,2,\ldots}$ in $L_2(\Gamma; p)$ by $\mathbf{G}^{(N)}$, the sufficient conditions for the CM to converge are obtained. The strong minimality of system Ψ in $L_2(\Gamma; p)$ has been proved [1].

Conditions (2.8) will be sufficient for the CM to converge. Consider the first limit of (2.8). Since $\delta B_{ij}^{(N)} = R_{ij}^{(N)}$, we have

$$\lim_{N\to\infty}\left\{\sum_{m=0}^{L}(N_m + N_{1m} + N_{2m})\sum_{m=0}^{L}[R_{mij}^{(N_m)} + R_{1mij}^{(N_{1m})} + R_{2mij}^{(N_{2m})}]\right\} = 0$$

We require

$$N_m = \xi_m N, \quad N_{1m} = \xi_{1m} N, \quad N_{2m} = \xi_{2m} N, \quad \sum_{m=0}^{L}(\xi_m + \xi_{1m} + \xi_{2m}) = 1 \tag{4.6}$$

Since the function $P(s)$ is infinitely-differentiable, and the functions $x_1(s)$, $x_2(s)$ are piecewise-linear along each integration segment, the first condition of (2.8) applies. If

$$\frac{d}{ds}[h(y(s))] \in \mathrm{Lip}([s_m, \theta_{1m}])$$
$$\frac{d}{ds}[h(y(s))] \in \mathrm{Lip}([\theta_{1m}, \theta_{2m}]) \tag{4.7}$$
$$\frac{d}{ds}[h(y(s))] \in \mathrm{Lip}([\theta_{2m}, s_{m+1}])$$

the second condition of (2.8) also applies. This yields the following theorem.

Theorem 2. If conditions (4.6) and (4.7) are satisfied, then

$$\lim_{N\to\infty}\|\bar{\mathbf{a}}^{(N)} - \tilde{\mathbf{a}}^{(N)}\|_2 = 0$$

It has been established that the CM converges. A similar comment to Remark 4 can be made regarding the condition number of the matrix of the system obtained by the collocation method.

The choice of the points θ_{rm} ($r = 1, 2, m = 0, 1, \ldots, L$) between segments on which the weighted and unweighted quadrature formulae are applied is important. It is better to choose them so that the integration errors on each of the segments $[s_m, \theta_{1m}]$, $[\theta_{1m}, \theta_{2m}]$, $[\theta_{2m}, s_{m+1}]$, $m = 0, 1, \ldots, L$ are of the same order. The estimate obtained on the basis of the Runge estimate in [1] is invalid, because it cannot be assumed, as the number N of basic functions increases, that the respective derivatives of the integrands on each integration segment are constant. This is because the basis functions of system Ψ have the form: $[x^2(s)+y^2(s)]^{k/2}\cos k\psi(s)$, $[x^2(s)+y^2(s)]^{k/2}$, where $k = 1, 2, \ldots, N/2$ and $\psi(s)$ is the angle between the radius vector of a point on the boundary $(x(s), y(s))$ and the Ox axis. As N increases, they will oscillate over the integration segments.

We define the points θ_{rm} $(r = 1, 2, m = 0, 1, \ldots, L)$ to give equal error estimates of the quadrature formulae in each of the intervals $[s_m, \theta_{1m}]$, $[\theta_{1m}, \theta_{2m}]$, $[\theta_{2m}, s_{m+1}]$ $(m = 0, 1, \ldots, L)$. We put $F(s) = \psi_{ij}(x_{1m}(s), x_{2m}(s))$ and consider the integral

$$\int\limits_{s_m}^{s_{m+1}} F(s)p(s)ds = \int\limits_{s_m}^{\theta_{1m}} F(s)p_{1m}(s)(s-s_m)^{\beta_m}ds + \int\limits_{\theta_{1m}}^{\theta_{2m}} F(s)p_m(s)ds +$$
$$+\int\limits_{\theta_{2m}}^{s_{m+1}} F(s)p_{2m}(s)(s_{m+1}-s)^{\beta_{m+1}}ds, \quad \beta_k = \alpha_k^{-1}-1, \quad k = m, m+1 \tag{4.8}$$

Let $R_{1m}^{N_{1m}}, R_m^{N_m}, R_{2m}^{N_{2m}}$ be the errors of the Gauss' quadrature formulae for the integrals on the right-hand side of (4.8), where the first and third integrals are found using weights $(s - s_m)^{\beta_m}$, $(s_{m+1} - s)^{\beta_m+1}$ of orders N_{1m} and N_{2m}, respectively, and the second using unweighted formula of order N_m. Then [7]

$$|R_m^{N_m}| \leq \frac{1}{(2N_m)!}\max_{s\in(\theta_{1m},\theta_{2m})}\left|\frac{d^{2N_m}}{ds^{2N_m}}[F(s)p(s)]\right|\int\limits_{\theta_{1m}}^{\theta_{2m}}\omega_{N_m}^2(s)ds \tag{4.9}$$

where $\omega_{Nm}(s)$ is the root polynomial of the unweighted Gauss quadrature formula. Calculating the derivative of order $2N_m$ of the integrand $F(s)p(s)$

$$\frac{d^{2N_m}}{ds^{2N_m}}[F(s)p(s)] = \sum_{j=0}^{2N_m}\binom{2N_m}{j}F^{(2N_m-j)}(s)p^{(j)}(s),$$

$$p^{(j)}(s) = \sum_{k=0}^{j}\binom{j}{k}p_m^{(j-k)}(s)\sum_{t=0}^{k}\binom{k}{t}(-1)^{k-t}\beta_m^{[t]}\beta_{m+1}^{[k-t]}(s-s_m)^{\beta_m-t}(s_{m+1}-s)^{\beta_{m+1}-k+t}$$

$$\beta_m^{[t]} = \beta_m(\beta_m-1)\cdot\ldots\cdot(\beta_m-t+1)$$

and also introducing the notation

$$p = \max_{m=0,1,\ldots,L}\sup_{\substack{s\in(s_m,s_{m+1})\\ j\in[0,2N_m]}}|p_m^{(j)}(s)|, \quad F = \sup_{\substack{s\in[0,1]\\ j\in[0,2N_m]}}|F^{(j)}(s)|$$

$$K_m = \left|\sum_{j=0}^{2N_m}\binom{2N_m}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{t=0}^{k}\binom{k}{t}\beta_m^{[t]}\beta_{m+1}^{[k-t]}\right|$$

from (4.9) we obtain

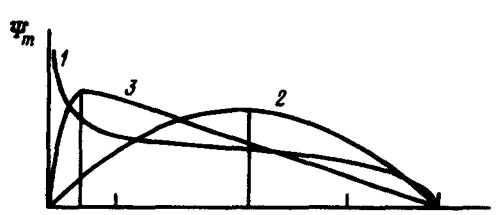

Fig. 1.

$$|R_m^{N_m}| \leqslant \frac{1}{(2N_m)!} pFK_m \max_{\lambda_1,\lambda_2 \in [0,2N_m]} \sup_{s \in (\theta_{1m},\theta_{2m})} \Psi_m(s,\lambda_1,\lambda_2)$$

$$\Psi_m(s,\lambda_1,\lambda_2) = (s - s_m)^{\beta_m - \lambda_1}(s_{m+1} - s)^{\beta_{m+1} - \lambda_2}$$

We introduce the notation

$$s_m^* = \frac{(\beta_{m+1} - \lambda_2)s_{m+1} + (\beta_m - \lambda_1)s_m}{\beta_m + \beta_{m+1} - \lambda_1 - \lambda_2}$$

and note that $(\Psi_m)'_s(s_m^*, \lambda_1, \lambda_2) = 0$ for $\lambda_1 + \lambda_2 \neq \beta_m + \beta_{m+1}$. Since the internal angles of the boundary Γ of the region Ω lie in the range $(0, 2\pi)$ we have $\alpha_m \in (0, 2)$ and $\beta_m \in (-1/2, +\infty)$.

We will consider the behaviour of the function Ψ_m in the segment $[\theta_{1m}, \theta_{2m}]$ and the position of the point s_m^* relative to points θ_{1m}, θ_{2m} in three cases (Fig. 1): when $\beta_m - \lambda_1 < 0$, $\beta_{m+1} - \lambda_2 > 0$ (curve 1), $s_m^* \in [\theta_{1m}, \theta_{2m}]$, $\beta_m - \lambda_1 > 0$, $\beta_{m+1} - \lambda_2 > 0$ (curve 2) and $s_m^* \notin [\theta_{1m}, \theta_{2m}]$, $\beta_m - \lambda_1 > 0$, $\beta_{m+1} - \lambda_2 > 0$ (curve 3).

For these cases, the estimate of the error when evaluating the second integral on the right-hand side of (4.8), using the unweighted Gauss quadrature formula, is

$$|R_m^{N_m}| \leqslant pFK_m T_m \chi_m(\theta_{1m},\theta_{2m})/(2N_m)! \tag{4.10}$$

$$\chi_m(\theta_{1m},\theta_{2m}) = \max\begin{cases} 1, & \text{for } \beta_m > 0,\ \beta_{m+1} > 0 \\ h_{1m}^{\beta_m - 2N_m}, & \text{for } \beta_{m+1} > 0,\ \beta_m - 2N_m < 0 \\ h_{2m}^{\beta_{m+1} - 2N_m}, & \text{for } \beta_m > 0,\ \beta_{m+1} - 2N_m < 0 \\ \sigma_m, & \text{for } \beta_m - 2N_m < 0,\ \beta_{m+1} - 2N_m < 0 \end{cases}$$

$$\sigma_m = \max\{h_{1m}^{\beta_m - 2N_m}(\bar{s}_m)^{\beta_{m+1} - 2N_m},\ h_{2m}^{\beta_{m+1} - 2N_m}(\bar{s}_m)^{\beta_m - 2N_m}\}$$

$$h_{1m} = \theta_{1m} - s_m, \quad h_{2m} = s_{m+1} - \theta_{2m}, \quad \bar{s}_m = (s_{m+1} - s_m)/2$$

$$T_m = (N_m)!^4[(2N_m)!^2(2N_m + 1)]^{-1}(s_{m+1} - s_m)^{2N_m + 1}$$

The error estimates for the first and third integrals on the right-hand side of (4.8) can be obtained in the same way.

By equating the resulting estimates, we obtain the points θ_{1m}, θ_{2m} between the segments of integration for the weighted and unweighted Gauss formulae. We can then find the nodes of the weighted and unweighted quadrature formulae and, using (4.5), obtain a CM with the stability property which gives a sequence of approximate solutions converging to the exact solution.

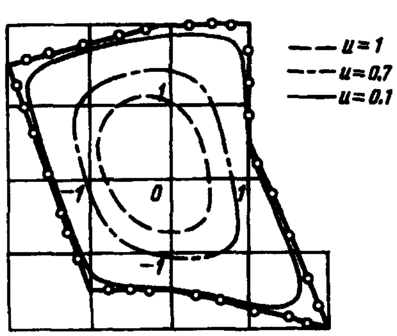

Fig. 2.

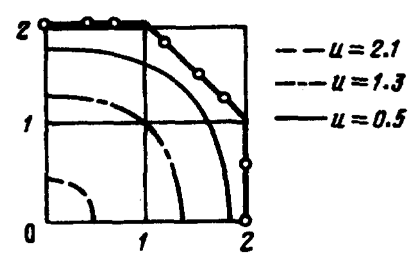

Fig. 3.

5. APPROXIMATE SOLUTION OF THE SAINT-VENANT PROBLEM OF THE TORSION OF ELASTIC PRISMS

Let Ω be a simply-connected region occupied by the section of a prism, and let the boundary $\Gamma = \partial\Omega$ be defined by (4.1). Solving the Saint-Venant problem reduces to finding a function u, which is harmonic in Ω, and on Γ satisfies the condition

$$u|_\Gamma(x_1,x_2)=(x_1^2+x_2^2)/2, \quad (x_1,x_2)\in\Gamma$$

We shall seek an approximate solution of the problem in the form of a finite series (1.2), where ψ_k are functions of system (3.1). We find $a_k^{(N)}$, the coefficients of the series (1.2), by applying to (2.10) the boundary analogue of the collocation method with weight functions (4.5). The vector $a_1^{(N)}, \ldots, a_N^{(N)}$ is a solution of (2.13).

The calculations were carried out for various polygonal regions. The level lines of the function u and the collocation points (denoted by small circles) are shown in Figs 2 and 3. Figure 3 shows only one-quarter of the region, in view of its symmetry. There are as many basis functions of system (3.1) as there are collocation points.

The calculations performed for a large number of basis functions (≈ 90) for the region shown in Fig. 3 gave no indication that the method was numerically unstable.

REFERENCES

1. ANUFRIYEV, I. Ye. and PETUKHOV, L. V., The solution of elliptic differential equations by expansion of the solution in terms of non-orthogonal basis functions. *Sbornik Trudov St. Petersburg, Goss. Tekh. Univ.*, 1996, 461, 120–129.
2. PETUKHOV, L. V. and ANUFRIYEV, I. Ye., The solution of elliptic differential equations by expansion of the solution in terms of non-orthogonal basis functions. In *Proceedings of the International Conference and Chebyshev Readings, to Celebrate the 175th Anniversary of the Birth of P. L. Chebyshev*. Vol. 2, 286–288. Izd. MGU, Moscow, 1996.
3. VOLKOV, Ye. A., *Block Method for Solving the Laplace Equation and for Constructing Conformal Mappings*. CRC Press, Boca Raton, 1994.
4. ALEKSIDZE, M. A., *The Solution of Boundary-value Problems by Expansion in Terms of Non-orthogonal Functions*. Nauka, Moscow, 1978.
5. MIKHLIN, S. G., *The Numerical Realization of Variational Methods*. Nauka, Moscow, 1966.
6. HORN, R. A. and JOHNSON, C. R., *Matrix Analysis*. Cambridge University Press, Cambridge, 1986.
7. BEREZIN, I. S. and ZHIDKOV, N. P., *Computational Methods*, Vol. 1. Nauka, Moscow, 1966.

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